

The University of Chicago

THE INEQUALITIES OF
MORSE FOR A PARAMETRIC PROBLEM OF THE CALCULUS
OF VARIATIONS

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1936

By

ROSS HARVEY BARDELL

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Reprinted from
CONTRIBUTIONS TO THE CALCULUS OF VARIATIONS, 1933-37
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Introduction. The problem of determining the relations satisfied by the numbers of extremal arcs of different types joining two fixed points for a problem of the calculus of variations in parametric form in the plane was considered by Morse (VI, page 213)¹ and Richmond (VII, page 248). The method of procedure in the paper by Morse is the application of results from an earlier paper (V, page 345) on the theory of critical points of real functions together with the theory of analysis situs. Richmond treats the problem by a more direct method under the original hypotheses of Morse. One of these hypotheses is the assumption that the region in which the extremals are sought is covered by a proper field of extremals. It is provable that the extremal arcs of the field can be transformed into a one-parameter family of parallel straight lines, and that the problem itself can consequently be given the non-parametric form. Another of the hypotheses of Morse is that the problem is reversible (VI, page 223). In a later paper (VIII, page 599) he has eliminated these assumptions, but his later method involves the use of somewhat complicated transformation theorems from analysis situs.

The object of the present paper is to prove inequalities analogous to those of Morse without the use of theorems from

¹The Roman numeral indicates a paper in the bibliography at the end of this paper.

analysis situs, and by a method quite different from that of Morse or Richmond. In the first four sections the proof is made under hypotheses comparable with the weaker assumptions of Morse. The results obtained in these sections are somewhat weaker than his under essentially the same hypotheses (VIII, page 623) in that they apply to so called sequences of adjacent extremals instead of individual extremals. Two extremals through the points 1 and 2 are said to be adjacent if there is no other extremal joining 1 with 2 and having an inclination at the point 1 between those of the two given ones. They are said to have the same type if they have on them the same number of points of contact with the envelope of the one-parameter family of extremals through the point 1. A sequence of extremals joining the points 1 and 2 is an ordered set of extremals joining these two points, with adjoining extremals of the set adjacent, and all having the same type which is different from the types of the preceding and following extremals adjacent to the set.

In the fifth section it is assumed in addition that the problem is reversible. The resulting inequalities are for individual extremals instead of sequences and are somewhat stronger than those obtained by Morse (VI, page 232; VIII, page 599) and Richmond (VII, page 251).

1. Analytic formulation of the problem. The following hypotheses are made concerning the integrand function $F(x, y, x', y')$ the region R in which F is defined, and the resulting family of extremal arcs.

(a) The integrand function $F(x, y, x', y')$ is an analytic function of its argument and satisfies the homogeneity condition

$$F(x, y, kx', ky') = kF(x, y, x', y') \quad (k > 0)$$

for every set (x, y, x', y') such that (x, y) is in a neighborhood of a region R of the xy -plane and (x', y') are any numbers such that

$$x'^2 + y'^2 \neq 0.$$

Further the function

$$F_1 = F_{x'x'}/y'^2 = -F_{x'y'}/x'y' = F_{y'y'}/x'^2$$

is positive for all sets (x, y, x', y') defined above.

(b) The region R is a portion of the xy -plane with a simply closed continuous boundary B consisting of a finite number of non-singular analytic arcs not tangent to each other at the corners of B . Further B is assumed to be extremal convex in the sense that an extremal arc which is tangent to B at a point P has no point interior to R in a neighborhood of P , while an extremal arc through a corner of B either crosses B at the corner or is such that it has no point interior to R in a neighborhood of the corner.

Under these assumption it can be shown (III, page 221) that through a point l interior to the region R there passes a one-parameter family of extremals

$$(1:1) \quad x = x(t, a), \quad y = y(t, a),$$

with the following property:

(1) The parameter t is the length of arc on each extremal (1:1) and at the initial value $t = 0$

$$(1:2) \quad \begin{aligned} x(0, a) &= x_1, & y(0, a) &= y_1, \\ x_t(0, a) &= \cos a, & y_t(0, a) &= \sin a, \end{aligned}$$

where (x_1, y_1) are the coordinates of the point l .

We shall assume the two further properties:

(2) On each extremal arc (1:1) there is a first point of intersection P with the boundary B defined by a parameter value $t = T(a)$. Further no point of the boundary B is a point envelope of the family (1:1).

(3) No one of the extremal arcs intersects itself on the interval $0 \leq t \leq T(a)$.

It is clear from the assumption of the convexity of the region R , that no extremal is tangent to the boundary at its value $T(a)$. From the assumption made upon the integrand function F it follows that the functions (1:1) are analytic in the variables t , a in a neighborhood of the region of points t , a defined by the conditions

$$0 \leq t \leq T(a), \quad 0 \leq a \leq 2\pi,$$

and are periodic in a with the period 2π . The function $T(a)$ is a part of a solution $T(a)$, $u(a)$ of the equations

$$(1:3) \quad x(t, a) = \xi(u), \quad y(t, a) = \eta(u),$$

where $\xi(u)$, $\eta(u)$ define the boundary B . Since no one of the extremals (1:1) can be tangent to B at the values $(t, a) = [T(a), a]$ the determinant

$$\begin{vmatrix} x_t(t, a) & \xi_u(u) \\ y_t(t, a) & \eta_u(u) \end{vmatrix}$$

is different from zero on the boundary B . Hence the solution $T(a)$, $u(a)$ of the equations (1:3) which is continuous for $0 \leq a \leq 2\pi$, is analytic at all values a except those which define corners of B . Furthermore $T(a)$ has the period 2π . The functional determinant

$$(1:4) \quad \Delta(t, a) = \begin{vmatrix} x_t(t, a) & x_a(t, a) \\ y_t(t, a) & y_a(t, a) \end{vmatrix}$$

has only isolated simple zeros when a is fixed since it is a solution of the Jacobi equation of the integral J and therefore can never vanish simultaneously with its derivative with respect to t without being identically zero (III, page 236). But it is not identically zero since one finds readily with the help of equation (1:2) that $\Delta_t(0, a) = 1$.

The points of contact of an extremal (1:1) with an envelope of the family are determined by the zeros of the determinant $\Delta(t, a)$ defined in (1:4) above (III, page 236). At every zero (t_0, a_0) of this determinant the derivative $\Delta_t(t, a)$ is different from zero, as was seen above. Hence there is always a solution $t = \tau(a)$ of the equation $\Delta(t, a) = 0$, defined and analytic in a neighborhood of $a = a_0$ and such that $\tau(a_0) = t_0$. It follows then that every point of contact of an extremal (1:1) with an envelope arc moves continuously along the extremal when the parameter a varies. It has been shown (I, page 322; III, page 360) that all analytic envelope arcs

$$(1:5) \quad x = x[\tau(a), a] = \bar{x}(a), \quad y = y[\tau(a), a] = \bar{y}(a),$$

thus defined have only four types of singularities. The arc may consist of a single point in which case $\bar{x}(a)$, $\bar{y}(a)$ are constants and the envelope arc reduces to a single point. If however $\bar{x}(a)$, $\bar{y}(a)$ are not constants then the only singular points possible are those at which the branches of the envelope are like those at an ordinary point and lie on the same side of the extremal tangent at the singular point, and cusps with branches extending either away from the point l or toward the point l , and separated by the

extremal tangent to the envelope at the singular point (I, page 323; III, page 360). In the third section of this paper it will be seen that under the assumptions made above there can be no envelope arc interior to R which is simply closed and has a continuously turning tangent. Every envelope arc in R must in fact begin and end on the boundary B .

2. Some fundamental lemmas. In 1912 Miles considered (IV, page 37) the problem of finding the number of extremal arcs joining two fixed points of a very special region R of the xy -plane, a neighborhood of an extremal arc whose ends are conjugate points. The method he employed was that of determining the point of intersection of an extremal arc E_a of the family through a point 1 with the ordinate through a point 2, and then showing that as the parameter a varied this point of intersection moved continuously along the ordinate, reversing its direction of motion at points of contact with an envelope arc of the family. That method is unsatisfactory for the purposes of the present paper since the region R defined in §1 (b) is more general than that of Miles and is not restricted to be a small neighborhood of a given initial extremal arc, but the fundamental idea of his proof can be utilized here again.

Consider an arc C in the region R defined by equations

$$(2:1) \quad x = u(s), \quad y = v(s),$$

where u and v are analytic and

$$u'^2 + v'^2 = 1$$

for all values of s on an interval $s' \leq s \leq s''$. Further suppose that C does not intersect itself, and is nowhere tangent to an arc of the family (1:1).

LEMMA 2:1. If the arc C defined by equations (2:1) has an intersection point with one of the arcs (1:1) then there exists single valued functions $s(a)$, $t(a)$ and an interval $a_1 \leq a \leq a_2$ of length $\leq 2\pi$ on which $s(a)$ and $t(a)$ are analytic and satisfy the equations

$$x[t(a), a] = u[s(a)], \quad y[t(a), a] = v[s(a)].$$

On the interval $a_1 a_2$ the function $s'(a)$ vanishes only at isolated values of a at which the values $t(a)$, a define a point on the envelope of the family (1:1), and $s'(a)$ changes sign only when $t(a)$, a defines a point on the envelope which is not a cusp. For values of a between a_1 and a_2 the point $u[s(a)]$, $v[s(a)]$ is an interior point of C while at each of the values $a = a_1$ and $a = a_2$ it coincides with an end value of C unless the length of the interval $a_1 a_2$ is 2π .

To prove this let the arc C be extended slightly so that the analytic properties of C are preserved on its extension. The equations

$$(2:2) \quad x(t, a) = u(s), \quad y(t, a) = v(s),$$

have by hypothesis an initial solution (a_0, s_0, t_0) at which

$$\begin{vmatrix} x_t(t, a) & u_s(s) \\ y_t(t, a) & v_s(s) \end{vmatrix} \neq 0.$$

Then from known existence theorems for implicit functions, equations (2:2) have an analytic solution $s(a)$, $t(a)$ in a neighborhood of $a = a_0$ and such that

$$s(a_0) = s_0, \quad t(a_0) = t_0.$$

Repetition of this argument, with an end value of the above neigh-

borhood and the corresponding values of $s(a)$, $t(a)$ as an initial solution, shows that the functions $s(a)$, $t(a)$ satisfying equations (2:2) are defined on a largest interval $a_1 \leq a \leq a_2$ of length $\leq 2\pi$ and containing a_0 , and such that for each a interior to this interval the corresponding intersection point is interior to C and for $a = a_1$ or $a = a_2$, where $a_2 - a_1 < 2\pi$, it is an end point of C . When $a_2 - a_1 = 2\pi$ the intersection points for $a = a_1$ and $a = a_1 + 2\pi$ may be either interior to the arc C or at an end point.

Differentiation of the relations (2:2) gives

$$(2:3) \quad \begin{aligned} u_s[s(a)]s'(a) - x_t[t(a), a]t'(a) &= x_a[t(a), a], \\ v_s[s(a)]s'(a) - y_t[t(a), a]t'(a) &= y_a[t(a), a]. \end{aligned}$$

Solving for $s'(a)$ we obtain

$$(2:4) \quad (x_tv_s - y_tu_s)s'(a) = x_ty_a - y_tx_a.$$

The right member of this equation is the function $\Delta[t(a), a]$ where $\Delta(t, a)$ is the determinant in (1:4) above which vanishes at all points on an envelope arc of the family (1:1) but is not identically zero (III, page 236). The function $\Delta[t(a), a]$ is not identically zero since the arc C is not an enveloping arc of the family (1:1). We see then that $s'(a)$ vanishes only for isolated values of a , and the values for which it vanishes define intersections of C with an envelope arc of the family (1:1).

The value of $s'(a)$ changes sign at a value a_3 when $\Delta[t(a), a]$ changes sign at that value, and this happens when the first non-vanishing derivative of $\Delta[t(a), a]$ at $a = a_3$ is

$$(2:5) \quad \frac{d^r}{da^r} \Delta[t(a), a]$$

with r odd. Then since the envelope is defined by the function

$\tau(a)$ which satisfies the equation $\Delta[\tau(a), a] \equiv 0$ (III, page 359) it follows that for the value a_3 the functions $t(a)$, $\tau(a)$ have their first $r - 1$ derivatives equal but

$$(2:6) \quad t^{(r)}(a_3) \neq \tau^{(r)}(a_3).$$

If we substitute the value of $s'(a)$ from (2:4) in the equations (2:3) we obtain

$$(2:7) \quad \begin{aligned} (x_t v_s - y_t u_s)(x_t t'(a) + x_a) &= u_s \Delta[t(a), a], \\ (x_t v_s - y_t u_s)(y_t t'(a) + y_a) &= v_s \Delta[t(a), a]. \end{aligned}$$

From equations (2:7) it follows that the first r derivatives with respect to a of the functions $x[t(a), a]$, $y[t(a), a]$ vanish at a_0 . The first $r - 1$ of these are equal to the first $r - 1$ derivatives of the functions $\bar{x}(a)$, $\bar{y}(a)$ defining the envelope, while the r -th derivatives are different. It follows that on the envelope are the first derivatives of the functions $\bar{x}(a)$, $\bar{y}(a)$ in (1:5) which do not both vanish at a_3 are the derivatives

$$(2:8) \quad \bar{x}^{(r)}(a_3), \quad \bar{y}^{(r)}(a_3).$$

Hence the equations of the envelope are take the form

$$(2:9) \quad \begin{aligned} \bar{x}(a) &= \bar{x}(a_3) + (a - a_3)^r [\bar{x}^{(r)}(a_3)]/r! + \dots, \\ \bar{y}(a) &= \bar{y}(a_3) + (a - a_3)^r [\bar{y}^{(r)}(a_3)]/r! + \dots, \end{aligned}$$

Since r is odd the envelope does not have a cusp at a_3 (III, page 360), and one sees that $s'(a)$ changes sign only at a point of intersection of C with the envelope of the family (1:1) which is not a cusp.

At a cusp of the envelope are the r in equations (2:9) is even (III, page (360). Then by the above reasoning it follows

that the r in relations (2:5) is also even and hence that $\Delta[t(a), a]$ does not change sign at $a = a_3$. As a consequence of relation (2:4) it follows that $s'(a)$ can not change sign at $a = a_3$.

A properly restricted segment of an extremal arc of the family (1:1) possesses all of the properties ascribed to the curve C in Lemma 2:1 above. Therefore the curve C may be replaced by a segment of an extremal arc of the family (1:1) and the following corollary results.

COROLLARY 2:1. Let E_{12} be a segment of an extremal arc E_{a_1} of the family (1:1) which is intersected by the arc E_a for every a on an interval $a' \leq a \leq a''$ which does not include a_1 . Then as a varies from a' to a'' each point of intersection P of E_a with E_{12} , as long as it remains on E_{12} , must move continuously along E_{12} , reversing its direction of motion only for those values of a for which $\Delta[t(a), a]$ changes sign, where t is the arc length measured along E_a . Further all points of intersection of E_a with E_{12} are distinct and must remain so for all values of a on the above interval.

The first statement of the corollary is an immediate consequence of Lemma 2:1 with the arc E_{12} replacing the arc C . The last statement follows as a result of the hypothesis that the arc $1P$ of E_a does not intersect itself, and of the fact that no two extremal arcs can be tangent unless they coincide.

LEMMA 2:2. If the arc l_2 of an extremal arc E_{a_1} of the family (1:1) is of type k , and if moreover the point 2 is not conjugate to 1 on E_{a_1} , then for values of a sufficiently near to a_1 the arc l_2 of E_{a_1} is intersected by E_a at exactly k points distinct from 1.

To prove this let the points of contact of E_{a_1} with the envelope be defined by the parameter values (t_i, a_i) where $i =$

1, 2, ..., k. Further let t'_1 and t''_1 where $t'_1 < t_1 < t''_1$ represent points on E_{a_1} near those defined by t_1 . Then for values of a sufficiently near a_1 the arc E_a will have no intersections with E_{a_1} on the intervals

$$t''_i < t_i < t'_{i+1} \quad (i = 1, 2, \dots, k),$$

where t is measured along E_{a_1} , and it is understood that t''_1 is zero, the value of t defining the point 1 on E_{a_1} , and t'_{k+1} is the value of t defining the point 2 on E_{a_1} . This follows since neighborhoods of the arcs of E_{a_1} defined on these intervals are simply covered by the extremal arcs E_a for a near to a_1 (III, page 249), and since the extremals simply cover a neighborhood of the point 1 with the exception of 1 itself (II, page 324).

Consider now a particular one of the points of contact of the arc l_2 of E_{a_1} with the envelope arc of family $(l:l)$. Let this point be designated by P_1 and the parameter values defining it by (t_1, a_1) . Let an xy -coordinate system be so chosen that with reference to it

$$x_t(t_1, a_1) \neq 0.$$

Then in a neighborhood of the point P_1 the extremal arc E_a can be represented by an equation of the form

$$y = y(x, a).$$

Further at each point of the envelope near P_1 it follows that

$$(2:10) \quad y_a(x, a) = 0, \quad y_{ax}(x, a) \neq 0,$$

since $y_a(x, a)$ is a solution of the Jacobi equation and can not vanish simultaneously with $y_{ax}(x, a)$. Since the function $y(x, a)$ is analytic the difference $y(x, a) - y(x, a_1)$ has the form

$$y(x, a) - y(x, a_1) = (a - a_1)y_a(x, a_1) + \dots$$

As a consequence of relations (2:10) there exists a $\delta > 0$ such that the sign of $y_a(x_1 - \delta, a_1)$ is different from that of $y_a(x_1 + \delta, a_1)$. Hence there exists an $\epsilon > 0$ such that for each a satisfying the relation

$$|a - a_1| \leq \epsilon$$

the difference $y(x, a) - y(x, a_1)$ vanishes at least once on the interval

$$(2:11) \quad x_1 - \delta < x < x_1 + \delta$$

From the relation

$$y_x(x, a) - y_x(x, a_1) = (a - a_1)y_{ax}(x, a_1) + \dots$$

it follows that $y(x, a) - y(x, a_1)$ is monotonic on the above interval and hence can vanish only once in a neighborhood of P_1 .

We can see this by taking δ so small that $y_{ax}(x, a_1) \neq 0$ on the interval (2:11) and taking ϵ so small that $y_x(x, a) - y_x(x, a_1)$ has the sign of $(a - a_1)y_{ax}(x, a_1)$ for

$$x_1 - \delta < x < x_1 + \delta, \quad |a - a_1| < \epsilon.$$

Hence in a neighborhood of each conjugate point the extremal arc E_{a_1} is intersected exactly once by each neighboring extremal E_a .

At an intersection of an extremal E_{a_1} with an extremal E_a let the parameter values on the two arcs be (t_1, a_1) and (t, a) . Since the arcs E_a and E_{a_1} are never tangent to each other we have the inequality

$$(2:12) \quad D(t_1, a_1; t, a) = \begin{vmatrix} x_t(t, a) & x_t(t_1, a_1) \\ y_t(t, a) & y_t(t_1, a_1) \end{vmatrix} = \sin(\theta_1 - \theta) \neq 0,$$

where θ and θ_1 are the inclinations of the tangents to E_a and E_{a_1} , respectively, at their intersection.

LEMMA 2:3. If E_{a_1} is an extremal arc of the family (1:1) which is of type $k > 0$, then for $a > a_1$ and near a_1 the sign of the determinant $D(t_1, a_1, t, a) = \sin(\theta_1 - \theta)$ at the i -th intersection following 1 of E_a with E_{a_1} is $(-1)^{i-1}$, and for $a < a_1$ and near a_1 it is $(-1)^i$, for $i = 1, 2, \dots, k$.

This is evident geometrically and can be established analytically with the help of a local coordinate system in a neighborhood of E_{a_1} simply covered by the normals to E_{a_1} .

LEMMA 2:4. For each fixed a the determinant $\Delta(t, a)$ is positive near the initial point 1 of E_a and has the sign $(-1)^i$ between the i -th and $(i + 1)$ -st points conjugate to 1 on E_a .

This follows at once from the facts, noted in the concluding paragraphs of Section 1 above, that $\Delta_t(0, a) = 1$ and that $\Delta_t(t, a)$ is different from zero at each point conjugate to 1 on E_a .

LEMMA 2:5. If an extremal E_{a_1} intersects an extremal E_{a_2} the order of the intersections on the two extremals is always the same.

To prove this suppose $a_1 < a_2$ and let the parameter a vary from a_1 to a_2 . For every value of $a \neq a_1$ the intersections of E_a and E_{a_1} are distinct since no two different extremals are ever tangent. For $a > a_1$ and near a_1 the intersections of E_a and E_{a_1} evidently have the same order on E_a and E_{a_1} . As a increases no intersection of E_a and E_{a_1} can move onto E_{a_1} at the point 1, since by hypothesis the arc E_a does not intersect itself. No new intersection of E_a and E_{a_1} can appear on E_{a_1} except at its intersection 4 with the boundary B , since no two different extremals are ever tangent. Let a' be the least upper bound of the values

a for which the order of the intersections on E_a and E_{a_1} is the same. The order must be the same on E_a and E_{a_1} since otherwise, by continuity considerations, it follows that for values $a < a'$ and near a' the order would also be different. For similar continuity reasons the order is the same on E_a and E_{a_1} for $a > a'$ and near a' . Hence there is no least upper bound a' of the kind described and the lemma is proved.

3. Properties of the extremal arcs joining two points of R .

A first lemma which will be useful is the following one:

LEMMA 3:1. Let 2 be a point of R or its boundary B , which is joined to 1 by at least one extremal arc E_{a_1} of the family $(1:1)$. Then the point 2 is also joined to 1 by an extremal arc E_{a_0} of the family $(1:1)$ such that the arc 12 of E_{a_0} has on it no point conjugate to 1 between 1 and 2.

Let the number of points conjugate to 1 between 1 and 2 on the arc E_{a_1} be designated by k_1 . If $k_1 = 0$ then E_{a_1} itself is an arc E_{a_0} of the type described in the lemma. If $k_1 > 0$ then as a consequence of Lemma 2:2 the extremal arc E_a , for $a > a_1$ and near a_1 , has k_1 or $k_1 + 1$ intersections with the arc 12 of E_{a_1} , the latter number being possible only when 2 is conjugate to 1 on E_{a_1} . It follows from Lemma 2:3 that as a increases from a_1 to $a_1 + 2\pi$ all of these intersections of E_a with the arc 12 of E_{a_1} must move off of the latter arc. Therefore on the interval $a_1 \leq a \leq a_1 + 2\pi$ there must exist a first value, say a_0 , such that the arc E_{a_0} has only the points 1 and 2 in common with the arc 12 of E_{a_1} . We shall prove that the arc 12 of E_{a_0} has on it no point conjugate to 1 between 1 and 2.

Let 3 be the first intersection on E_{a_1} of the arc E_a with E_{a_1} for $a_1 < a < a_0$. The arc 13 of E_a has no intersection with E_{a_1} other than 1 and 3. Otherwise there would be a least upper

bound a' of the values a for which this is true. The arc 13 of E_a , could not cross E_{a_1} elsewhere than at 1 and 3, since otherwise a' would not be the least upper bound just described; and it could not be tangent to E_{a_1} anywhere since no two different extremals are ever tangent in R . Hence the arc between 1 and 3 of E_a is always distinct from E_{a_1} .

For $a < a_0$ and near a_0 the arc 13 of E_a has at least k_0 intersections with the arc 12 of E_{a_0} , by Lemma 2:2, where k_0 is the number of points conjugate to 1 between 1 and 2 on the arc E_{a_0} . As a decreases to a_1 these intersections can not pass off of the arc 12 of E_{a_0} , since 3 remains on the arc 12 of E_{a_1} and, as seen above, the arc 13 of E_a , for $a_1 < a < a_0$ is distinct from E_{a_1} . Consequently the arc E_{a_1} must have at least k_0 intersections with E_{a_0} between 1 and 2. This is, however, impossible unless $k_0 = 0$ on account of the definition of a_0 .

COROLLARY 3:1. Let 2 be a point of R on its boundary B which is not on an envelope arc of the family (1:1), and which is joined to 1 by at least one extremal arc of this family. Then the point 2 is also joined to 1 by an extremal arc E_{a_0} of the family (1:1) such that the arc 12 of E_{a_0} is of type zero.

This is an immediate consequence of Lemma 3:1.

COROLLARY 3:2. Every envelope arc of the family (1:1), lying in the region R , must begin and end on the boundary B of R .

Suppose the corollary not true. Then there would be a closed envelope arc

$$(3:1) \quad x = \bar{x}(a), \quad y = \bar{y}(a) \quad (a' \leq a \leq a' + 2\pi)$$

of the type defined in equations (1:5) which has no point in common with the boundary B of R . Each extremal E_a of the family (1:1) would then have at least one point on it which is interior

to R and conjugate to l . This is impossible by Lemma 3:1 applied to an arc of E_a extending from l to the boundary B of R .

LEMMA 3:2. Every point of the region R is joined to l by at least one extremal arc of the family $(1:1)$.

To prove this we note first that there are at most a finite number of pairs (t, a) defining intersection points of the boundary B with the envelope of family $(1:1)$. Otherwise there would be an infinite number of triples (t, a, u) satisfying the equations

$$(3:2) \quad x(t, a) = \xi(u), \quad y(t, a) = \eta(u), \quad \Delta(t, a) = 0$$

where $\xi(u)$ and $\eta(u)$ define the boundary B . Hence there would exist a limiting triple (t_1, a_1, u_1) satisfying equations (3:2) and such that in every neighborhood of it there is an infinite number of other triples which also satisfy (3:2). This is impossible since the solutions of the equation $\Delta(t, a) = 0$ near (t_1, a_1) are all given by the function $t = \tau(a)$ which defines an envelope arc not tangent to the boundary B and hence having only one point in common with B .

There must therefore be a point 2 of B joined to l by an extremal of the family $(1:1)$ and such that 2 is not on an envelope arc of this family. By Corollary 3:1 there is an extremal arc E_{a_0} joining l and 2 and such that the arc $l2$ of E_{a_0} is of type zero.

Let the region R be cut along the extremal arc E_{a_0} from the point l , and designate by R' the region so obtained. The boundary B' of R' consists of the arc $l2$ of E_{a_0} , the boundary B of R , and the arc $2l$ of E_{a_0} . Consider now the subregion R'_a of R' which is bounded by the arc lP of E_a , where P is the first intersection of E_a with B' , and by the arc lP of B' . Suppose

that 3 is a point of R which is not joined to 1 by an extremal arc of the family $(1:1)$. For $a > a_0$ and near a_0 the point 3 is outside of R'_a . For $a < a_0 + 2\pi$ and near $a_0 + 2\pi$ it is not outside R'_a . Let a' be the least upper bound of the values $a > a_0$ for which 3 is not in R'_a . Then it is easy to see that 3 must be on the arc $E_{a'}$, contrary to hypothesis, and the lemma is therefore established.

COROLLARY 3:3. Every point 2 of R , which is not on an envelope arc of the family $(1:1)$, is joined to 1 by at least one extremal arc E_{a_0} of family $(1:1)$ such that the arc 12 of E_{a_0} is of type zero.

This is an immediate consequence of Corollary 3:1 and Lemma 3:2.

LEMMA 3:3. Every point 2 of R or its boundary B , which is not on an envelope arc of the family $(1:1)$, is joined to 1 by $2r + 1$ extremal arcs of the family $(1:1)$ of which r are of odd type and $r + 1$ of even type.

To prove this we note first that the point 2 is joined to 1 by a finite number of extremal arcs of the family $(1:1)$. Otherwise there would be an infinite number of pairs (t, a) defining arcs which join these points. Hence there would exist a limiting pair (t_1, a_1) which would define an arc joining 1 and 2 such that in every neighborhood of (t_1, a_1) there would be an infinite number of pairs (t, a) also defining arcs joining 1 and 2. This is impossible since by hypothesis $\Delta(t_1, a_1) \neq 0$ and therefore there would be a neighborhood of (t_1, a_1) in which there could be no pairs defining arcs joining 1 and 2.

As a consequence of Corollary 3:3 the point 2 is joined to 1 by an arc E_{a_0} such that the arc 12 of E_{a_0} is of type zero. Then the arc E_a , for $a > a_0$ and near a_0 , and also for $a < a_0 + 2\pi$

and near $a_0 + 2\pi$, has no intersections with the arc 12 of E_{a_0} . Therefore as a increases from a_0 to $a_0 + 2\pi$ every intersection of E_a with the arc 12 of E_{a_0} which moves on to this arc must move off again. Further an intersection which moves onto the arc 12 of E_{a_0} with $\Delta(t_2, a) > 0$ (< 0) moves off from that arc with $\Delta(t_2, a) < 0$ (> 0), since in the formula (2:4) applied to the arcs E_a and 12 of E_{a_0} , in place of E_a and C , the coefficient of $s'(a)$ never changes sign at an intersection of E_a and the arc 12 of E_{a_0} , and the motion of the intersection is in opposite directions only for opposite signs of $\Delta(t, a)$. Furthermore, as a consequence of Lemma 2:4 the number of conjugate points on E_a preceding the point 2 is odd or even according as $\Delta(t_2, a)$ is negative or positive, so that in the set of arcs joining the points 1 and 2 the number of arcs of even type, exclusive of E_{a_0} , must equal the number of arcs of odd type. This complete the proof.

Two extremal arcs, $E_{a'}$ and $E_{a''}$, joining the points 1 and 2 are said to be adjacent if there is no extremal arc E_a joining 1 and 2 and having a parameter value a between a' and a'' .

LEMMA 3:4. Adjacent arcs in the set of extremal arcs joining the points 1 and 2, where 2 is not on an envelope arc of the family (1:1) are such that the arcs 12 of each are either both of the same type or have types differing by unity.

To prove this let $E_{a'}$ and $E_{a''}$, where $a' < a''$, be two adjacent arcs in the set of extremal arcs joining the points 1 and 2. Let k' designate the type of the arc 12 of $E_{a'}$, and k'' the type of the arc 12 of $E_{a''}$. Then by Lemma 2:2 the arc E_a , for $a > a'$ and near a' , has k' intersections with the arc 12 of $E_{a'}$. Since by definition $E_{a'}$ and $E_{a''}$ are adjacent no intersections of E_a with the arc 12 of $E_{a'}$ can move off or on this latter arc

through the point 2, and consequently their number can not change as a increases from a' to a'' . At a'' either a new intersection of E_a with the arc 12 of $E_{a'}$, moves onto the arc 12 of $E_{a'}$, or the last one of the k' intersections of these arcs moves off of the arc 12 of $E_{a'}$. Hence $E_{a''}$ has k' or $k' + 1$ intersections with the arc 12 of $E_{a'}$, including the intersection at 2 but not the one at 1. By the same reasoning it follows, by allowing a to decrease from a'' to a' , that $E_{a'}$ has k'' or $k'' + 1$ intersections with the arc 12 of $E_{a''}$, including the intersection at 2 but not the one at 1. Therefore one of the following relations must hold

$$k' = k'', \quad k' = k'' \pm 1.$$

A sequence of arcs of the set of extremal arcs joining the points 1 and 2 is a set of adjacent arcs of the set all having the same type k and such that the arcs preceding and following the set have types different from k . Two sequences are adjacent if the initial arc of one is adjacent to the final arc of the other.

COROLLARY 3:4. Adjacent sequences of arcs in the set of extremal arcs joining the points 1 and 2, differ in type by unity.

This is an immediate consequence of Lemma 3:4 and the definitions of sequences and adjacent sequences given above.

LEMMA 3:5. Let 2 be a point of R not on an envelope arc of the family of extremals through 1. If the set of extremal arcs joining 1 to 2 contains more than one arc then it must contain at least two arcs $E_{a'}$ and $E_{a''}$ which are adjacent and such that the arc 12 of each is of type zero.

The point 2 is joined to 1 by an extremal arc E_{a_0} such that the arc 12 of E_{a_0} is of type zero, by Corollary 3:3. Designate by $a_1, \dots, a_{2r}$ the values on $a_0 < a < a_0 + 2\pi$ for which E_a

passes through the point 2. There is an even number of these arcs as a consequence of Lemma 3:3. If either the arc 12 of E_{a_1} or the arc 12 of $E_{a_{2r}}$ is of type zero then Lemma 3:5 is true. The only other possibility is that both of these arcs be of type one, by Lemma 3:4, and this case can be treated as follows.

Let 3 denote always the first intersection of E_a with the arc 12 of E_{a_0} . We prove first that there exists a first value a'' in the set $a_1, \dots, a_{2r}$ such that as a increases through a'' the point 3 is moving onto the arc 12 of E_{a_0} with $\sin(\theta_0 - \theta'') < 0$, where θ_0 and θ'' are the inclination of the tangents to E_{a_0} and $E_{a''}$, respectively, at their intersection 2. Formulas (2:4) and (2:12) applied to the parameter value $t(a)$ on E_{a_0} of an intersection of E_a and the arc 12 of E_{a_0} give

$$(3:3) \quad t'(a) = \Delta(t, a) / \sin(\theta_0 - \theta),$$

where now θ_0 and θ are the inclinations of the tangents to E_{a_0} and E_a , respectively, at the intersection. The sign of $\sin(\theta_0 - \theta)$ never changes as long as the intersection remains on the arc 12 of E_{a_0} since E_a and E_{a_0} are nowhere tangent. Since the arc 12 of E_{a_0} is of type zero the intersection 3 must move off of it through the point 2 as a increases through the value a_{2r} where $\Delta(t, a) < 0$ since the arc 12 of $E_{a_{2r}}$ is by hypothesis of type one. From (3:3) we see then that $\sin(\theta_0 - \theta_{2r}) < 0$ and consequently that $\sin(\theta_0 - \theta_1) < 0$ at the intersection 2 of E_{a_0} and E_{a_1} at which the point 3 last moved onto the arc 12 of E_{a_0} . Hence there is a first value a'' in the set $a_1, \dots, a_{2r}$ at which 3 is moving onto the arc 12 of E_{a_0} with $\sin(\theta_0 - \theta'') < 0$. At all of the values of a preceding a'' for which 3 moves onto the arc 12 of E_{a_0} we must also have $\sin(\theta_0 - \theta) > 0$, since this inequality holds at all values at which 3 moves off of the arc 12 of E_{a_0} .

As a increases from a_0 to a'' the arc E_a either has no intersections with the arc 12 of $E_{a''}$ or else its intersections with the arc 12 of $E_{a''}$ are all preceded by a first intersection 3 of E_a with the arc 12 of E_{a_0} . Consider first, values $a_0 < a < a_1$. As a increases from a_0 the single intersection 2 of E_a with the arc 12 of $E_{a''}$ is moving off of the latter arc, by a formula analogous to (3:3) with a_0 replaced by a'' , and since $\Delta(t, a) > 0$ at 2 on E_{a_0} and $\sin(\theta'' - \theta_0) > 0$. Hence for $a_0 < a < a_1$ the arc E_a has no intersections with the arc 12 of $E_{a''}$, or with the arc 12 of E_{a_0} since the latter is, by hypothesis, of type zero.

At $a = a_1$ we have a first intersection 3 of E_a moving onto the arc 12 of E_{a_0} with $\sin(\theta_0 - \theta_1) > 0$, and since $\sin(\theta_0 - \theta'') < 0$ we must have one or the other of the configurations shown in Figures 1 and 2. The second is impossible since

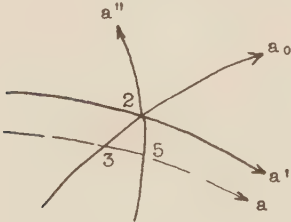


Figure 1

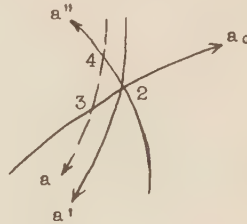


Figure 2

in that case as 3 moves onto the arc 12 of E_{a_0} an intersection 4 of E_a with the arc 12 of $E_{a''}$ would have to move off of the arc 12 of $E_{a''}$ and there is no such intersection. Hence for $a > a_1$ and near a_1 the arc E_a has a first intersection 3 with the arc 12 of E_{a_0} , and the arc 13 of E_a has no intersections with $E_{a''}$.

As long as 3 stays on the arc 12 of E_{a_0} with increasing a the arc 13 of E_a has no intersection anywhere on $E_{a''}$, since a first such intersection would have to be a point of tangency of E_a and $E_{a''}$ and that is impossible. At the first value a_1 following a_0 at which the intersection 3 moves off of the arc 12 of E_{a_0} we must again have one of the configurations in Figures 1 and 2 with a_1 replaced by a_1 . The second configuration would imply the existence of an intersection 4 of the arc 13 of E_a with $E_{a''}$ for $a < a_1$ which is impossible as seen above. Figure 1 then shows that as the intersection 3 moves off of the arc 12 of E_{a_0} the first intersection 5 on E_a of E_a with $E_{a''}$ moves off of the arc 12 of $E_{a''}$. The point 5 is also the first intersection on $E_{a''}$ of E_a with $E_{a''}$, by Lemma 2:5. Thus we see that the arc E_a for $a > a_1$ and near a_1 has no intersection with the arc 12 of $E_{a''}$. Repetition of this argument for succeeding values a_1 where 3 moves on or off of the arc 12 of E_{a_0} shows the truth of the last italicized statement above.

The arcs 12 of $E_{a''}$ and the adjacent extremal E_{a_1} with $a' < a''$ are both of type zero. This is evident by Lemma 2:2 for the arc 12 of $E_{a''}$ since E_{a_1} has no intersections with that arc for $a < a''$ and near a'' . The arc 12 of E_{a_1} is of type zero or one, by Lemma 3:4. In formula (3:3) we have $t'(a') > 0$, $\sin(\theta_0 - \theta') > 0$, and hence $\Delta(t_2, a') > 0$ which shows that the arc 12 of E_{a_1} must be of even type and therefore of type zero. Thus the Lemma 3:5 is proved.

4. The inequalities for sequences. Let the extremal $E_{a''}$ of Lemma 3:5 above be designated by the initial parameter value a_0 . Then the extremal arc E_{a_1} of the same lemma is the one with the largest parameter value in the set on the interval $a_0 \leq a < a_0 + 2\pi$ designating extremals of the family through 1 which also

pass through 2. As a consequence of Corollary 3:4 it follows that the set of extremal arcs joining the points 1 and 2 contains an odd number of sequences of such arcs, the first and last such sequence being of type zero, and adjacent sequences in the set differing in type by unity. Let the sequences after the first one be paired into distinct consecutive pairs, that is, the second with the third, fourth with the fifth, etc.

LEMMA 4:1. If p_i represents the number of pairs of sequences of extremal arcs joining the points 1 and 2, which contain sequences of type i and $i - 1$, then $p_i \geq 1$ ($i = 1, \dots, n$) where n is the maximum of the type numbers of the extremal arcs joining the points 1 and 2.

There are two cases to be considered, namely, i odd and i even.

When i is odd, then by the definition above of the pairing of the sequences the first sequence in each pair is of odd type. Hence when i is odd the sequence of type i is the first sequence in a pair to which it belongs. Then the last sequence of type i in the set of extremal arcs joining the points 1 and 2 must be paired with a sequence of type $i - 1$. Otherwise, since adjacent sequences differ in type by unity, it would be paired with a sequence of type $i + 1$. In that event it could not be the last sequence of type i since the last sequence in the set of arcs joining the points 1 and 2 is of type zero and adjacent sequences differ in type by unity. Hence in this case $p_i \geq 1$.

When i is even the sequence of type i must be the second sequence in a pair to which it belongs. Then by exactly the same argument that was used above it follows that the first sequence of type i in the set of extremal arcs joining the points 1 and 2 must be paired with a sequence of type $i - 1$. Hence $p_i \geq 1$ in

this case also and the lemma is proved.

Let S_i represent the number of sequences of type i in the set of extremal arcs joining the points 1 and 2, and further let n be the maximum type of any sequence in this set. Then one finds readily that

$$\begin{aligned}
 S_0 &= 1 + p_1, \\
 S_1 &= p_1 + p_2, \\
 &\dots \\
 (4:1) \quad S_i &= p_i + p_{i+1}, \\
 &\dots \\
 S_n &= p_n.
 \end{aligned}$$

Elimination of the p_i 's by subtraction yields the relations

$$\begin{aligned}
 S_0 &= 1 + p_1, \\
 S_1 &= S_0 - 1 + p_2, \\
 &\dots \\
 (4:2) \quad S_i &= S_{i-1} - S_{i-2} + \dots + (-1)^{i-1} S_0 + (-1)^i + p_{i+1}, \\
 &\dots \\
 S_n &= S_{n-1} - S_{n-2} + \dots + (-1)^{n-1} S_0 + (-1)^n.
 \end{aligned}$$

Now as a consequence of Lemma 4:1 relations (4:2) become

$$\begin{aligned}
 S_0 &> 1, \\
 S_1 &> S_0 - 1, \\
 &\dots \\
 (4:3) \quad S_i &> S_{i-1} - S_{i-2} + \dots + (-1)^{i-1} S_0 + (-1)^i, \\
 &\dots \\
 S_n &= S_{n-1} - S_{n-2} + \dots + (-1)^{n-1} S_0 + (-1)^n.
 \end{aligned}$$

Summarizing the results of the preceding pages we have

THEOREM 4:1. For a calculus of variations problem and a region R satisfying the hypotheses made in Section 1 above, let 1 and 2 represent two fixed points of R such that 2 is not on an envelope arc of the family of extremal arcs through 1. Further let S_1 represent the number of sequences of extremal arcs of type 1 joining the points 1 and 2, and let n be the maximum type of any such sequence. Then the numbers S_1 ($i = 0, 1, \dots, n$) so defined must satisfy the relation (4:3).

When the maximum type n is greater than zero the application of Lemma 4:1 to equations (4:1) gives

$$(4:4) \quad \begin{aligned} S_1 &\geq 2 & (i = 0, 1, \dots, n-1), \\ S_n &\geq 1. \end{aligned}$$

Let M_1 represent the number of extremal arcs joining the points 1 and 2 on each of which the arc 12 is of type i. Then obviously $M_1 \geq S_1$ and hence the relations (4:4) imply

$$\begin{aligned} M_1 &\geq 2 & (i = 0, 1, \dots, n-1), \\ M_n &\geq 1. \end{aligned}$$

5. The inequalities for extremals when the problem is reversible. In place of the positive homogeneity postulated in Section 1 it is assumed in this section that for every (x, y) in R and all x' and y' such that

$$x'^2 + y'^2 \neq 0$$

the stronger homogeneity property

$$(5:1) \quad F(x, y, kx', ky') = |k|F(x, y, x', y')$$

holds for all values of k, positive or negative. All of the other hypotheses made in Section 1 are assumed to hold here also.

Under these assumptions the differential equations for the family of extremal arcs through the point 1 can be put in the form (VI, page 223)

$$(5:2) \quad \frac{dx}{ds} = \cos \tau, \quad \frac{dy}{ds} = \sin \tau, \quad \frac{d\tau}{ds} = (F_{yx'} - F_{xy'})/F_1,$$

where the arguments of F_1 and the derivatives of F are $x, y, \cos \tau, \sin \tau$. It follows then from relation (5:1) and its consequences for the derivative of F that if $x(s), y(s), \tau(s)$ is a solution of (5:2) then

$$\bar{x}(s) = x(-s), \quad \bar{y}(s) = y(-s), \quad \bar{\tau}(s) = \tau(-s) + \pi$$

is also a solution. A problem having this property is said to be reversible (VI, page 223), i.e. the extremal arcs through the initial values x_0, y_0, τ_0 and $x_0, y_0, \tau_0 + \pi$ are identical.

The lemmas and corollaries of Sections 2 and 3 hold here also and in addition we can now prove the following lemma and corollary.

LEMMA 5:1. Let 2 be a point of the region R which is not on an envelope arc of the family of extremals through the point 1. Then adjacent extremal arcs joining the points 1 and 2 and having the same type must be of type zero. It follows at once that there are no sequences of type $k > 0$ containing more than one arc.

To prove this consider two extremal arcs $E_{a'}$ and $E_{a''}$ ($a' < a''$) joining the points 1 and 2 which are adjacent and of the same type k' . Suppose $k' > 0$. Then by lemma 2:2 the extremal arc E_{a_1} for $a > a_1$ and near a_1 , has k' intersections with the arc 12 of $E_{a'}$. Let the last of these k' intersections of $E_{a'}$ with the arc 12 of E_{a_1} be designated by 3 and the intersection of $E_{a'}$ with the boundary B by P. Further let the points of intersection of

$E_{a'}$ with the boundary B on opposite sides of l be designated by 4 and 5 so that the points 5 1 3 2 4 appear in that order on $E_{a'}$.

Since $E_{a'}$ and $E_{a''}$ are adjacent it follows that as a increases from a' to a'' the number of intersections of the arc l_1 of $E_{a'}$ with the arc l_2 of $E_{a'}$ can not change. As a increases through a'' the intersection 3 of $E_{a'}$ with the arc l_2 of $E_{a'}$ can not move off of the arc l_2 of $E_{a'}$. This follows since by Lemma 2:3 the sign of $\sin(\theta' - \theta)$ at the intersection point 3 is $(-1)^{k'-1}$, and since for an arc $E_{a''}$ of type k' the sign of $\Delta(t_2, a'')$ at the point 2 is $(-1)^{k'}$, so that the quotient $\Delta(t, a)/\sin(\theta' - \theta)$ in (3:3) would be negative if the point 3 were at 2 and the point 3 could not be moving off of the arc l_2 of $E_{a'}$ for a increasing through a'' . It is evident then that a new intersection point 6 of $E_{a'}$ with the arc l_2 of $E_{a'}$ must be moving on to the latter arc at the point 2 as a increases through the value a'' . Since the sign of $\Delta(t_2, a'')$ at the point 2 is $(-1)^{k'}$ and the point 6 is moving onto the arc l_2 of $E_{a'}$ as a increases through a'' , it follows that the quotient $\Delta(t, a)/\sin(\theta' - \theta)$ in (3:3) is negative and therefore that the sign of $\sin(\theta' - \theta)$ at the intersection 6 is $(-1)^{k'-1}$.

There will be a last value a'' on the interval $a' < a < a''$ for which the point 6 is at 4. This follows because 6 is the intersection of $E_{a'}$ with $E_{a'}$, next following 3 and the signs of $\sin(\theta' - \theta)$ are the same at 6 and 3 as was seen in the last paragraph. For a near to a' the sign of $\sin(\theta' - \theta)$ at an intersection adjacent to 3 must be $(-1)^{k'}$, by Lemma 2:3, and hence the intersection of $E_{a'}$ with $E_{a'}$, next following 3 for $a > a'$ and near a' must move off and 6 with its sign $(-1)^{k'-1}$ for $\sin(\theta' - \theta)$ must move on to the arc l_4 of $E_{a'}$ for some value a'' as stated.

The arc 54 of E_a , divides the region R in two parts. It was seen above that the arc E_a'' crosses E_a with the same sign for $\sin(\theta' - \theta)$ at 3 and 2. Therefore the arc 32 of E_a'' must cross the arc 54 of E_a . But the arc 32 of E_a'' can not cross the arc 12 of E_a , since by definition 3 is the last intersection of E_a' and E_a'' on the arc 12 of E_a , and hence also the last on the arc 12 of E_a'' . Finally the arc 32 of E_a'' can not cross the arc 24 of E_a' . This follows since, as was seen above, the point of intersection 6 of E_a'' with E_a , must move from 2 off of the arc 15 of E_a' for a decreasing from a'' to a' . Therefore every point of intersection 7 of the arc 32 of E_a'' with the arc 24 of E_a' , must likewise move off of E_a' . Hence there would be a first value $\bar{a}$ preceding a'' on the interval $a' < a < a''$ such that the arc 36 of E_a'' would have contact with the boundary B of R . But this is impossible since the region R is extremal convex. It is evident then that the arc 32 of E_a'' must cross the arc 51 of E_a , at one point 7 at least.

Consider now the family of extremal arcs through the point 2. The arcs E_a' and E_a'' defined above belong to this family since the problem is assumed to be reversible. Let these arcs be defined in the family through 2 by the parameter values b_1 and b_2 , respectively. Then the arc 21 of E_{b_2} intersects the arc 25 of E_{b_1} in the points 3 and 7 at least, and these points are in the order 2 3 7 on E_{b_1} . As was seen above $\sin(\theta' - \theta'')$ has the same sign at the intersections 2 and 3 of E_{b_2} and E_{b_1} . Since by Lemma 2:3 the sign of $\sin(\theta' - \theta)$ alternates at consecutive intersections of E_b with E_{b_1} for b near b_1 , it follows that as b varies from b_2 to b_1 the intersection 3 must move off of the arc 25 of E_{b_1} . Since 7 is on the arc 35 of E_{b_1} it follows that 7 must move off of E_{b_1} ahead of 3. Hence there exists a first value b_3 for

which the arc 23 of E_{b_3} has contact with the boundary B , and this is impossible on account of the extremal convexity of the region R . Therefore the assumption that $k' > 0$ leads to a contradiction and the lemma is proved.

COROLLARY 5:1. Let 2 be a point of the region R which is not on an envelope arc of the family of extremals through 1. Then if the set of extremal arcs joining the points 1 and 2 contains more than one arc it contains exactly one pair of arcs which are adjacent and both of the same type zero.

To prove this we recall that as a consequence of Lemma 3:3 the number of extremal arcs joining the points 1 and 2 is odd. Furthermore from Lemma 3:5 we know that the set of extremal arcs joining 1 and 2 contains at least one pair of extremal arcs, $E_{a'}$ and $E_{a''}$, which are adjacent and of the same type zero. Let the set of extremal arcs joining the points 1 and 2 now be arranged in order of their parameter values with arcs $E_{a''}$ and $E_{a'}$ at the beginning and end of the set, respectively. The arcs of the set beginning with $E_{a''}$ can be grouped into distinct consecutive pairs such that each arc except $E_{a'}$ appears in exactly one such pair. By Lemmas 3:4 and 5:1 it follows that the arcs of each pair either differ in type by unity or are of the same type zero. Let p be the number of pairs in which the arcs differ in type by unity, and let q be the number of pairs in which the arcs are both of type zero. Then there are p arcs of odd type and $p + 2q + 1$ arcs of even type. Since by Lemma 3:3 the number of extremal arcs joining 1 and 2 of even type exceeds the number of odd type by unity it follows that $q = 0$.

Now let the extremal arcs joining the points 1 and 2 be arranged in distinct consecutive pairs with $E_{a''}$ left over. Then each arc except $E_{a''}$ will appear in exactly one such pair. By a

repetition of the argument used in the last paragraph above we see again that there are no pairs in which the extremal arcs are both of type zero. From these two results it follows that the arcs E_{a_1} and E_{a_n} are the only extremal arcs joining the points 1 and 2 which are adjacent and both of type zero. This completes the proof of Corollary 5:1.

With the initial parameter chosen as in the preceding paragraph, or as in Section 4, it follows as a consequence of Lemma 5:1 and Corollary 5:1 that every sequence contains but a single arc and consequently that

$$M_1 = S_1 \quad (i = 0, 1, 2, \dots, n),$$

where M_1 is as above, the number of extremal arcs of type i joining the points 1 and 2. Hence the inequalities (4:3) become

$$\begin{aligned} M_0 &> 1, \\ M_1 &> M_0 - 1, \\ &\dots \\ (5:3) \quad M_1 &> M_{1-1} - M_{1-2} + \dots + (-1)^{1-1} M_0 + (-1)^1, \\ &\dots \\ M_n &= M_{n-1} - M_{n-2} + \dots + (-1)^{n-1} M_0 + (-1)^n. \end{aligned}$$

Therefore Theorem 4:1 can be restated in terms of the M_1 as follows.

THEOREM 5:1. For a calculus of variations problem and a region R satisfying the hypotheses made in Section 1 above together with the stronger homogeneity assumption implying reversibility made at the beginning of this section, let 1 and 2 represent two fixed points of R such that 2 is not on an envelope arc of the family of extremal arcs through 1. Further let M_1 represent the number of extremal arcs joining the points 1 and 2

and each of type i , and let n be the maximum type of any such arc.
Then the numbers M_i ($i = 0, 1, 2, \dots, n$) so defined must satisfy
relations (5:3).

It should be noted that the conditions (5:3) are stronger than the results obtained by Morse (VI, page 232) and Richmond (VII, page 257) since the equality sign is missing except in the last equation.

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